Decomposing the SWAP quantum gate

2006 J. Phys. A: Math. Gen. 39 1463

(http://iopscience.iop.org/0305-4470/39/6/018)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 1463–1467 doi:10.1088/0305-4470/39/6/018

Decomposing the SWAP quantum gate

Y Hardy and W-H Steeb

International School for Scientific Computing, University of Johannesburg, Auckland Park 2006, South Africa

E-mail: yha@na.rau.ac.za

Received 6 October 2005, in final form 28 November 2005
Published 25 January 2006
Online at stacks.iop.org/JPhysA/39/1463

Abstract

Quantum gates are represented by unitary operators on a Hilbert space. We consider the unitary operators on $\mathcal{H}\otimes\mathbf{C}^2$ and describe the component operators on each Hilbert space in the product space. From these criteria we derive a decomposition for a specific class of unitary operators such that each operator in the product has Schmidt rank bounded by 2. This decomposition directly yields the XOR (controlled NOT) implementation of the SWAP operation.

PACS numbers: 03.65.Db, 03.67.−a

1. Introduction

Quantum gates are represented by unitary operators on a Hilbert space [1–3]. For completeness, we present the definitions of the Schmidt rank for states and operators below.

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two finite dimensional Hilbert spaces with the underlying field $\mathbf{C}$. Let $|\psi\rangle$ denote a pure state in the Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B$.

The *Schmidt number* (also called the *Schmidt rank*) of $|\psi\rangle\in\mathcal{H}_A\otimes\mathcal{H}_B$ over $\mathcal{H}_A\otimes\mathcal{H}_B$ is the smallest non-negative integer $\mathrm{Sch}(|\psi\rangle,\mathcal{H}_A,\mathcal{H}_B)$ such that $|\psi\rangle$ can be written as

$$|\psi\rangle=\sum_{j=1}^{\mathrm{Sch}(|\psi\rangle,\mathcal{H}_A,\mathcal{H}_B)}|\psi_j\rangle_A\otimes|\psi_j\rangle_B$$

where $|\psi_j\rangle_A\in\mathcal{H}_A$ and $|\psi_j\rangle_B\in\mathcal{H}_B$. Consequently for

$$|\psi\rangle=\sum_{j=1}^{\dim\mathcal{H}_A}\sum_{k=1}^{\dim\mathcal{H}_B}\psi_{jk}|j\rangle_A\otimes|k\rangle_B,$$

where the $|j_A\rangle$ and $|k_A\rangle$ form an orthonormal basis in their respective Hilbert spaces, we find that

$$\mathrm{Sch}(|\psi\rangle,\mathcal{H}_A,\mathcal{H}_B)=\mathrm{rank}(\psi_{jk})$$

where (ψ_{jk}) denotes the $\dim\mathcal{H}_A\times\dim\mathcal{H}_B$ matrix with entries ψ_{jk}.

The Hilbert–Schmidt inner product of two linear operators A and B acting on the same Hilbert space $\mathcal{H}$ is given by $(A, B) := \mathrm{Tr}(B^* A)$, where Tr denotes the trace. Thus the linear operators form a Hilbert space with the Hilbert–Schmidt inner product. Consequently the definition of the Schmidt rank for pure states can also be applied to matrices (linear operators).

The Schmidt rank of a linear operator $L : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$ over $\mathcal{H}_A \otimes \mathcal{H}_B$ is the smallest non-negative integer $\mathrm{Sch}(L, \mathcal{H}_A, \mathcal{H}_B)$ such that L can be written as

$$L = \sum_{j=1}^{\mathrm{Sch}(L,\mathcal{H}_A,\mathcal{H}_B)} L_{j,A} \otimes L_{j,B}$$

where $L_{j,A} : \mathcal{H}_A \to \mathcal{H}_A$ and $L_{j,B} : \mathcal{H}_B \to \mathcal{H}_B$ are linear operators.

Nielsen *et al* found that operators could be written in the *operator-Schmidt decomposition* [5]:

$$L = \sum_{j=1}^{\mathrm{Sch}(L,\mathcal{H}_A,\mathcal{H}_B)} s_j L_{j,A} \otimes L_{j,B}, \qquad s_j \neq 0 \quad j = 1, 2, \ldots, \mathrm{Sch}(L, \mathcal{H}_A, \mathcal{H}_B)$$

$$(L_{j,A}, L_{k,A}) = (L_{j,B}, L_{k,B}) = \delta_{jk}, \qquad j, k = 1, 2, \ldots, \mathrm{Sch}(L, \mathcal{H}_A, \mathcal{H}_B).$$

This decomposition yields more information about the operator L due to the orthogonality of the operators $L_{j,A}$ and $L_{j,B}$.

For an orthonormal basis $\{|0\rangle, |1\rangle\}$ in $\mathbf{C}^2$ we define the unitary operators:

$$U_{\mathrm{NOT}} := |0\rangle\langle 1| + |1\rangle\langle 0|$$
$$U_{\mathrm{NOT}}(1) := U_{\mathrm{NOT}} \otimes I_2$$
$$U_{\mathrm{NOT}}(2) := I_2 \otimes U_{\mathrm{NOT}}$$
$$U_{\mathrm{CNOT}}(1, 2) := |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes U_{\mathrm{NOT}}$$
$$U_{\mathrm{CNOT}}(2, 1) := I_2 \otimes |0\rangle\langle 0| + U_{\mathrm{NOT}} \otimes |1\rangle\langle 1|$$
$$U_{\mathrm{SWAP}} := |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|.$$

Of course, U_{SWAP} is composed of three U_{CNOT} operations:

$$U_{\mathrm{SWAP}} = U_{\mathrm{CNOT}}(2, 1) U_{\mathrm{CNOT}}(1, 2) U_{\mathrm{CNOT}}(2, 1).$$

This is the XOR implementation of the SWAP operation. This type of decomposition is useful in applications such as communication complexity where it is clear that information must be communicated in both directions to achieve the SWAP operation, whereas the CNOT operation only requires communication in one direction. Thus it would be useful to find a decomposition that illustrates this structure.

2. Operators on $\mathcal{H} \otimes \mathbf{C}^2$

Here we consider operators on $\mathcal{H} \otimes \mathbf{C}^2$ where $\mathcal{H}$ is an arbitrary finite dimensional Hilbert space with dimension n. Let U_q be a unitary operator on $\mathcal{H} \otimes \mathbf{C}^2$. We can write U_q in the form

$$U_q = Q_0 \otimes |0\rangle\langle 0| + Q_1 \otimes |0\rangle\langle 1| + Q_2 \otimes |1\rangle\langle 0| + Q_3 \otimes |1\rangle\langle 1|$$

where Q_0, Q_1, Q_2 and Q_3 are linear operators in $\mathcal{H}$ (some of which may be the zero operator), and $\{|0\rangle, |1\rangle\}$ is an orthonormal basis in $\mathbf{C}^2$. Next we determine the constraints on Q_0, Q_1, Q_2 and Q_3 from the condition $U_q U_q^* = U_q^* U_q = I_n \otimes I_2$. Since U_q is a unitary operator, we have

$$\begin{aligned} U_q U_q^* &= (Q_0 Q_0^* + Q_1 Q_1^*) \otimes |0\rangle\langle 0| + (Q_0 Q_2^* + Q_1 Q_3^*) \otimes |0\rangle\langle 1| \\ &\quad + (Q_2 Q_2^* + Q_3 Q_3^*) \otimes |1\rangle\langle 1| + (Q_2 Q_0^* + Q_3 Q_1^*) \otimes |1\rangle\langle 0| \\ &= I_n \otimes I_2 = U_q^* U_q. \end{aligned}$$

I_n denotes the $n \times n$ identity operator and 0_n denotes the $n \times n$ zero operator. Thus we find the conditions on the operators Q_j:

$$\begin{aligned} Q_0Q_0^* + Q_1Q_1^* &= I_n & \qquad Q_0^*Q_0 + Q_2^*Q_2 &= I_n \\ Q_2Q_2^* + Q_3Q_3^* &= I_n & \qquad Q_1^*Q_1 + Q_3^*Q_3 &= I_n \\ Q_2Q_0^* + Q_3Q_1^* &= 0_n & \qquad Q_1^*Q_0 + Q_3^*Q_2 &= 0_n. \end{aligned} \tag{1}$$

The operators Q_0, Q_1, Q_2 and Q_3 can be written in polar form $Q_j = U_jH_j$, where U_j is unitary and H_j is positive semi-definite for $j = 0, 1, 2, 3$. Inserting the polar form into equations (1) yields the set of equations

$$\begin{aligned} &H_0^2 + H_2^2 = I_n \\ &H_0H_2 = -H_2H_0(U_3^*U_2)^*(U_1^*U_0) \\ &H_1 = (U_3^*U_2)H_2(U_3^*U_2)^* = (U_1^*U_0)H_2(U_1^*U_0)^* \\ &H_3 = (U_3^*U_2)H_0(U_3^*U_2)^* = (U_1^*U_0)H_0(U_1^*U_0)^*. \end{aligned}$$

We note that if $Q_1 = 0_n$ ($H_1 = 0_n$) then $Q_2 = 0_n$. Similarly $Q_2 = 0_n$ implies $Q_1 = 0_n$ and $Q_0 = 0_n \iff Q_3 = 0_n$. In other words, when one of Q_0, Q_1, Q_2 and Q_3 is the zero operator, the non-zero operators of Q_0, Q_1, Q_2 and Q_3 are unitary. This result is independent of the chosen basis $\{|0\rangle, |1\rangle\}$. Thus we find that the Schmidt rank of U_q over $\mathcal{H} \otimes \mathbf{C}^2$ is either 1, 2 or 4. (This result was previously demonstrated by Dür *et al* [4] and Nielsen *et al* [5].)

3. Structure

To illustrate the structure further we suppose none of the operators Q_j are unitary (Schmidt rank 4 over $\mathcal{H} \otimes \mathbf{C}^2$); in other words, the operators Q_0, Q_1, Q_2 and Q_3 are linearly independent. The operators Q_0, Q_1, Q_2 and Q_3 span a four-dimensional vector space which is isomorphic to the space spanned by $\{|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|\}$. The operators $\{|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|\}$ are linearly independent, with the matrix representations in the standard basis $\{|0\rangle, |1\rangle\}$ of $\mathbf{C}^2$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The vector space isomorphisms which do not violate the conditions (1) are as follows:

(1) $Q_0 \to |0\rangle\langle 0|$, $Q_1 \to |1\rangle\langle 0|$, $Q_2 \to |0\rangle\langle 1|$, $Q_3 \to |1\rangle\langle 1|$
$\to U_{\text{SWAP}}$
(2) $Q_0 \to |1\rangle\langle 0|$, $Q_1 \to |0\rangle\langle 0|$, $Q_2 \to |1\rangle\langle 1|$, $Q_3 \to |0\rangle\langle 1|$
$\to U_{\text{NOT}}(1)U_{\text{SWAP}}$
(3) $Q_0 \to |0\rangle\langle 1|$, $Q_1 \to |1\rangle\langle 1|$, $Q_2 \to |0\rangle\langle 0|$, $Q_3 \to |1\rangle\langle 0|$
$\to U_{\text{NOT}}(2)U_{\text{SWAP}}$
(4) $Q_0 \to |1\rangle\langle 1|$, $Q_1 \to |0\rangle\langle 1|$, $Q_2 \to |1\rangle\langle 0|$, $Q_3 \to |0\rangle\langle 0|$
$\to U_{\text{NOT}}(1)U_{\text{NOT}}(2)U_{\text{SWAP}}$
(5) $Q_0 \to |0\rangle\langle 0|$, $Q_1 \to |1\rangle\langle 1|$, $Q_2 \to |0\rangle\langle 1|$, $Q_3 \to |1\rangle\langle 0|$
$\to U_{\text{CNOT}}(1, 2)U_{\text{SWAP}}$
(6) $Q_0 \to |1\rangle\langle 0|$, $Q_1 \to |0\rangle\langle 1|$, $Q_2 \to |1\rangle\langle 1|$, $Q_3 \to |0\rangle\langle 0|$
$\to U_{\text{NOT}}(1)U_{\text{CNOT}}(1, 2)U_{\text{SWAP}}$
(7) $Q_0 \to |0\rangle\langle 1|$, $Q_1 \to |1\rangle\langle 0|$, $Q_2 \to |0\rangle\langle 0|$, $Q_3 \to |1\rangle\langle 1|$
$\to U_{\text{NOT}}(2)U_{\text{CNOT}}(1, 2)U_{\text{SWAP}}$

(8) $Q_0 \to |1\rangle\langle 1|, \quad Q_1 \to |0\rangle\langle 0|, \quad Q_2 \to |1\rangle\langle 0|, \quad Q_3 \to |0\rangle\langle 1|$
$\to U_{\rm NOT}(1)U_{\rm NOT}(2)U_{\rm CNOT}(1,2)U_{\rm SWAP}$
(9) $Q_0 \to |0\rangle\langle 0|, \quad Q_1 \to |1\rangle\langle 0|, \quad Q_2 \to |1\rangle\langle 1|, \quad Q_3 \to |0\rangle\langle 1|$
$\to U_{\rm CNOT}(2,1)U_{\rm SWAP}$
(10) $Q_0 \to |1\rangle\langle 0|, \quad Q_1 \to |0\rangle\langle 0|, \quad Q_2 \to |0\rangle\langle 1|, \quad Q_3 \to |1\rangle\langle 1|$
$\to U_{\rm NOT}(1)U_{\rm CNOT}(2,1)U_{\rm SWAP}$
(11) $Q_0 \to |1\rangle\langle 1|, \quad Q_1 \to |0\rangle\langle 1|, \quad Q_2 \to |0\rangle\langle 0|, \quad Q_3 \to |1\rangle\langle 0|$
$\to U_{\rm NOT}(2)U_{\rm CNOT}(2,1)U_{\rm SWAP}$
(12) $Q_0 \to |0\rangle\langle 1|, \quad Q_1 \to |1\rangle\langle 1|, \quad Q_2 \to |1\rangle\langle 0|, \quad Q_3 \to |0\rangle\langle 0|.$
$\to U_{\rm NOT}(1)U_{\rm NOT}(2)U_{\rm CNOT}(2,1)U_{\rm SWAP}.$

The only non-local unitary operators are described in terms of $U_{\rm SWAP}$ or $U_{\rm CNOT}$. Dür *et al* similarly found that 2 qubit operations could be classified in terms of equivalence classes characterized by local unitary, $U_{\rm SWAP}$ and $U_{\rm CNOT}$ operations [4]. This structure is apparent in a restricted class of unitary operators described below.

4. Decomposing the SWAP gate

Now consider the restricted class of unitary operators with $Q_j = U_j\Pi_j$ where U_j is unitary and Π_j is a projection operator for $j = 0, 1, 2, 3$. This is a special case of the polar decomposition of Q_j. Inserting these assumptions into equations (1) yields

$$\Pi_0 + \Pi_2 = I_n$$
$$\Pi_1 = (U_3^* U_2)\Pi_2(U_3^* U_2)^* = (U_1^* U_0)\Pi_2(U_1^* U_0)^*$$
$$\Pi_3 = (U_3^* U_2)\Pi_0(U_3^* U_2)^* = (U_1^* U_0)\Pi_0(U_1^* U_0)^*.$$

Consequently we also find

$$\Pi_0\Pi_2 = \Pi_0(I_n - \Pi_0) = 0_n.$$

Thus Π_0 and Π_2 decompose the Hilbert space $\mathcal{H}$ into two orthogonal subspaces. Additionally, Π_1 and Π_3 also decompose $\mathcal{H}$ into two orthogonal subspaces and is equivalent to the decomposition given by $\{\Pi_0, \Pi_2\}$ under the unitary transformation $U_1^* U_0$ (or $U_3^* U_2$). This allows us to decompose U_q as a product of three operators each of Schmidt rank not greater than 2:

$$\begin{aligned} U_q &= U_0\Pi_0 \otimes |0\rangle\langle 0| + U_1\Pi_1 \otimes |0\rangle\langle 1| + U_2\Pi_2 \otimes |1\rangle\langle 0| + U_3\Pi_3 \otimes |1\rangle\langle 1| \\ &= U_0\Pi_0 \otimes |0\rangle\langle 0| + U_0\Pi_2 U_0^* U_1 \otimes |0\rangle\langle 1| + U_2\Pi_2 \otimes |1\rangle\langle 0| + U_2\Pi_0 U_2^* U_3 \otimes |1\rangle\langle 1| \\ &= (U_0 \otimes |0\rangle\langle 0| + U_2 \otimes |1\rangle\langle 1|)(\Pi_0 \otimes I_2 + \Pi_2 \otimes U_{\rm NOT}) \\ &\quad \times (I_n \otimes |0\rangle\langle 0| + [\Pi_0 U_2^* U_3 + \Pi_2 U_0^* U_1] \otimes |1\rangle\langle 1|). \end{aligned}$$

The operator $\Pi_0 U_2^* U_3 + \Pi_2 U_0^* U_1$ is obviously unitary. The decomposition has the interpretation of an exchange of information (1 qubit) from the Hilbert space $\mathbf{C}^2$ described by the projection operators $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and the Hilbert space $\mathcal{H}$ described by the projection operators $\{\Pi_0, \Pi_2\}$.

The operator $U_{\rm SWAP}$ has $U_0 = U_3 = I_2, U_1 = U_2 = U_{\rm NOT}, \Pi_0 = \Pi_1 = |0\rangle\langle 0|$ and $\Pi_2 = \Pi_3 = |1\rangle\langle 1|$. Consequently

$$\begin{aligned} U_{\rm SWAP} &= (I_2 \otimes |0\rangle\langle 0| + U_{\rm NOT} \otimes |1\rangle\langle 1|)(|0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes U_{\rm NOT}) \\ &\quad \times (I \otimes |0\rangle\langle 0| + [|0\rangle\langle 0|U_{\rm NOT} + |1\rangle\langle 1|U_{\rm NOT}] \otimes |1\rangle\langle 1|) \\ &= U_{\rm CNOT}(2,1)U_{\rm CNOT}(1,2)U_{\rm CNOT}(2,1); \end{aligned}$$

that is, we derived the XOR implementation of SWAP.

References

[1] Steeb W-H 1998 *Hilbert Spaces, Wavelets, Generalised Functions and Modern Quantum Mechanics* (Dordrecht: Kluwer)

[2] Steeb W-H and Hardy Y 2006 *Problems and Solutions in Quantum Computing and Quantum Information* 2nd edn (Singapore: World Scientific)

[3] Nielsen M A and Chuang I L 2002 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[4] Dür W, Vidal G and Cirac J I 2002 Optimal conversion of nonlocal unitary operations *Phys. Rev. Lett.* **89** 057901

[5] Nielsen M A, Dawson C M, Dodd J L, Gilchrist A, Mortimer D, Osborne T J, Bremner M J, Harrow A W and Hines A 2003 Quantum dynamics as a physical resource *Phys. Rev.* A **67** 052301